

# Hardy inequality and fractional Leibnitz rule for perturbed Hamiltonians on the line

Vladimir Georgiev,

Department of Mathematics, University of Pisa, Largo B. Pontecorvo 5,  
Pisa, 56127 Italy,  
georgiev@dm.unipi.it

Anna Rita Giammetta,

Department of Mathematics, University of Pisa, Largo B. Pontecorvo 5,  
Pisa, 56127 Italy,  
giammetta@mail.dm.unipi.it

June 29, 2016

## Abstract

We consider the following perturbed Hamiltonian  $\mathcal{H} = -\partial_x^2 + V(x)$  on the real line. The potential  $V(x)$ , satisfies a short range assumption of type

$$(1 + |x|)^\gamma V(x) \in L^1(\mathbb{R}), \quad \gamma > 1.$$

We study the equivalence of classical homogeneous Sobolev type spaces  $\dot{H}_p^s(\mathbb{R})$ ,  $p \in (1, \infty)$  and the corresponding perturbed homogeneous Sobolev spaces associated with the perturbed Hamiltonian. It is shown that the assumption zero is not a resonance guarantees that the perturbed and unperturbed homogeneous Sobolev norms of order  $s = \gamma - 1 \in [0, 1/p)$  are equivalent. As a corollary, the corresponding wave operators leave classical homogeneous Sobolev spaces of order  $s \in [0, 1/p)$  invariant.

**Keywords:** Homogeneous Sobolev norms, Paley Littlewood decomposition, Elliptic estimates, Laplace operator with potential, Equivalent Sobolev norms.

## 1 Introduction and motivation

The uncertainty principle in quantum mechanics is frequently associated with Hardy type inequality

$$\| |x|^{-s} f \|_{L^p(\mathbb{R}^n)} \leq C \| \mathcal{H}_0^{s/2} f \|_{L^p(\mathbb{R}^n)}, \quad s \in [0, n/p), \quad (1.1)$$

where  $\mathcal{H}_0 = -\Delta$  is the free Hamiltonian in  $\mathbb{R}^n$ ,  $n \geq 1$ . The presence of a perturbed Hamiltonian  $\mathcal{H} = \mathcal{H}_0 + V(x)$  with a short range real-valued potential  $V(x)$  leads to the natural question to verify if Hardy type inequality is true for this perturbed Hamiltonian. The appearance of eigenvectors of  $\mathcal{H}$  is an obstacle to have Hardy type inequality or to establish existence and completeness of the wave operators in the whole  $L^p(\mathbb{R}^n)$  space, so it is natural to look for estimate of type

$$\| |x|^{-s} f \|_{L^p(\mathbb{R}^n)} \leq C \| \mathcal{H}_{ac}^{s/2} f \|_{L^p(\mathbb{R}^n)}, \quad s \in [0, n/p), \quad (1.2)$$

where  $\mathcal{H}_{ac}$  is the absolutely continuous part of the perturbed Hamiltonian and  $f$  is in the domain of  $\mathcal{H}_{ac}$ .

Our key goal in this work is to study the equivalence of the fractional energy norms

$$\| \mathcal{H}_{ac}^{s/2} f \|_{L^p(\mathbb{R})} \sim \| \mathcal{H}_0^{s/2} f \|_{L^p(\mathbb{R})}, \quad (1.3)$$

since this equivalence property shows that (1.1) implies (1.2).

Another motivation to study the equivalence property (1.3) is connected with the necessity to generalize so called fractional Leibnitz rule, used as a basic tool in rigorous analysis of local well-posedness of nonlinear dispersive equations, to the case of fractional Hamiltonians of type  $\mathcal{H}_{ac}^{s/2}$ . To be more precise, the following estimate is known as fractional Leibnitz rule or Kato-Ponce estimate (one can see [9] for the proof)

$$\|\mathcal{H}_0^{s/2}(fg)\|_{L^p(\mathbb{R})} \leq C\|\mathcal{H}_0^{s/2}f\|_{L^{p_1}(\mathbb{R})}\|g\|_{L^{p_2}(\mathbb{R})} + C\|f\|_{L^{p_3}(\mathbb{R})}\|\mathcal{H}_0^{s/2}g\|_{L^{p_4}(\mathbb{R})}, \quad (1.4)$$

where the parameters  $s, p, p_j, j = 1, \dots, 4$ , satisfy

$$s > 0, \quad 1 < p, p_1, p_2, p_3, p_4 < \infty, \quad \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}.$$

The estimate can be considered as natural homogeneous version of the non-homogeneous inequality of type (1.4) involving Bessel potentials  $(1 - \mathcal{H}_0)^{s/2}$  in the place of  $\mathcal{H}_0^{s/2}$ , obtained by Kato and Ponce in [13] (for this the estimates of type (1.4) are called Kato-Ponce estimates, too). More general domain for parameters can be found in [8]. A more precise estimate can be deduced when  $0 < s < 1$ . More precisely, Kenig, Ponce, and Vega [14] obtained the estimate

$$\|\mathcal{H}_0^{s/2}(fg) - f\mathcal{H}_0^{s/2}g - g\mathcal{H}_0^{s/2}f\|_{L^p(\mathbb{R})} \leq C\|\mathcal{H}_0^{s_1/2}f\|_{L^{p_1}(\mathbb{R})}\|\mathcal{H}_0^{s_2/2}g\|_{L^{p_2}(\mathbb{R})}, \quad (1.5)$$

provided

$$0 < s = s_1 + s_2 < 1, \quad s_1, s_2 \geq 0,$$

and

$$1 < p, p_1, p_2 < \infty, \quad \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}. \quad (1.6)$$

Therefore, one can pose the question to find appropriate short range assumptions on the perturbed Hamiltonian so that the fractional Leibnitz rule (1.4) or the more precise bilinear estimate (1.5) are valid for this perturbed Hamiltonian. Since the equivalence property (1.3) implies (1.4), it is important to determine admissible domain for the parameters  $s > 0, p \in (1, \infty)$ , where (1.3) holds. The uncertainty principle restriction  $s < 1/p$  is a reasonable candidate and we aim at studying if this is the optimal domain where (1.3) is fulfilled.

We can make another interpretation of (1.3) connecting  $\|\mathcal{H}_0^{s/2}f\|_{L^p(\mathbb{R})}$  with the homogeneous Sobolev spaces  $\dot{H}_p^s(\mathbb{R})$  and observing that (1.3) guarantees the invariance of the action of the wave operators

$$W_{\pm} = s - \lim_{t \rightarrow \pm\infty} P_{ac}(\mathcal{H})e^{it\mathcal{H}}e^{-it\mathcal{H}_0}$$

on these homogeneous Sobolev spaces.

The existence and completeness of the wave operators in standard Hilbert space (typically Lebesgue space  $L^2$ ) in case of short range perturbations is well known (see [15], [16], [12] and the references therein). The functional calculus for the absolutely continuous part  $\mathcal{H}_{ac} = P_{ac}(\mathcal{H})\mathcal{H}$  of the perturbed non-negative operator  $\mathcal{H}$  can be introduced with a relation involving  $W_{\pm}$

$$g(\mathcal{H}_{ac}) = W_+g(\mathcal{H}_0)W_+^* = W_-g(\mathcal{H}_0)W_-^*, \quad (1.7)$$

for any function  $g \in L_{loc}^{\infty}(0, \infty)$ . Moreover, the wave operators map unperturbed Sobolev spaces in the perturbed ones,

$$W_{\pm} : D(\mathcal{H}_0^{s/2}) \rightarrow D(\mathcal{H}_{ac}^{s/2})$$

and we have

$$W_{\pm} : \dot{H}_p^s(\mathbb{R}) \rightarrow \dot{H}_{p, \mathcal{H}_{ac}}^s(\mathbb{R}), \quad \forall s \geq 0, \quad 1 < p < \infty,$$

where  $\dot{H}_{p, \mathcal{H}_{ac}}^s(\mathbb{R})$  is the perturbed homogeneous Sobolev space generated by the Hamiltonian  $\mathcal{H}_{ac}$ . More precisely,  $\dot{H}_{p, \mathcal{H}_{ac}}^s(\mathbb{R})$  is the homogeneous Sobolev spaces associated with the absolutely continuous part  $\mathcal{H}_{ac}$

of the perturbed Hamiltonian  $\mathcal{H} = \mathcal{H}_0 + V$ . This is the closure of functions  $f \in S(\mathbb{R})$  orthogonal<sup>1</sup> to the eigenvectors of  $\mathcal{H}$  with respect to the norm

$$\|f\|_{\dot{H}_{p,\mathcal{H}_{ac}}^s(\mathbb{R})} = \left\| \mathcal{H}_{ac}^{s/2} f \right\|_{L^p(\mathbb{R})}. \quad (1.8)$$

The equivalence property (1.3) implies that the homogeneous Sobolev space  $\dot{H}_p^s(\mathbb{R})$  is invariant under the action of the wave operators  $W_{\pm}$  for  $0 \leq s < 1/p$ .

## 2 Assumptions and main results

The study of the dispersive properties of the evolution flow in some cases of short range perturbed Hamiltonians  $\mathcal{H}$  shows (see [2], [7]) that homogeneous Sobolev norms for perturbed and unperturbed Hamiltonians are equivalent

$$\|\mathcal{H}_{ac}^{s/2} f\|_{L^2(\mathbb{R}^n)} \sim \|\mathcal{H}_0^{s/2} f\|_{L^2(\mathbb{R}^n)}, \quad (2.1)$$

provided  $s < n/2$ . Our goal is to extend this equivalence to the case

$$\|\mathcal{H}_{ac}^{s/2} f\|_{L^p(\mathbb{R}^n)} \sim \|\mathcal{H}_0^{s/2} f\|_{L^p(\mathbb{R}^n)}, \quad (2.2)$$

with  $s < n/p$ .

First, we shall show that the requirement  $s < n/p$  is optimal, i.e. we shall prove the following result:

**Theorem 1.** *If  $n \geq 1$  and  $V(x)$  is defined as follows*

$$V(x) = \frac{1}{1 + |x|^3}, \quad (2.3)$$

*then (1.2) with  $s = n/p \leq 2$  is not true.*

Our next goal is to obtain (1.2) in the admissible range  $s \in [0, n/p)$  for the case  $n = 1$ . First we shall describe the assumptions on the potential  $V$ .

We shall assume that the potential  $V : \mathbb{R} \rightarrow \mathbb{R}$  is a real-valued potential,  $V \in L^1(\mathbb{R})$  and  $V$  is decaying sufficiently rapidly at infinity, namely following [18] we require

$$\|\langle x \rangle^\gamma V\|_{L^1(\mathbb{R})} < \infty, \quad \gamma \geq 1, \quad (2.4)$$

or equivalently we assume  $V \in L_\gamma^1(\mathbb{R})$ , where

$$L_\gamma^1(\mathbb{R}) = \{f \in L_{loc}^1(\mathbb{R}); \langle x \rangle^\gamma f(x) \in L^1(\mathbb{R})\}, \quad \langle x \rangle^2 = 1 + x^2.$$

Our key assumption on  $V$  is that zero is not a resonance point. The precise definition of the notion of resonance point at the origin is given in Definition 4.4 by the aid of the relation

$$T(0) = 0.$$

The point spectrum of  $\mathcal{H}$  consists of real numbers  $\lambda \in (-\infty, 0]$ , such that

$$\mathcal{H}f - \lambda f = 0, \quad f \in L^2(\mathbb{R}), \quad (2.5)$$

and absolutely continuous part  $[0, \infty)$ . We shall denote by  $L_{pp}^2(\mathbb{R})$  the linear space generated by the eigenvectors  $f$  in (2.5). This is finite dimensional space and its orthogonal complement in  $L^2$  is the invariant subspace, where the perturbed Hamiltonian  $\mathcal{H}$  is absolutely continuous.

The key tool to prove the Hardy inequality and the fractional Leibnitz rule (1.5) is the following estimate.

---

<sup>1</sup> the precise definition of eigenvectors is given below in (2.5)

**Theorem 2.** *Suppose*

$$V \in L^1_\gamma(\mathbb{R}), \quad \gamma > 1, \quad s = \gamma - 1 < 1/p, \quad p \in (1, \infty)$$

*and the perturbed Hamiltonian  $\mathcal{H}$  has no resonance at the origin. Then there exists a positive constant  $C = C(s, p) > 0$  so that we have*

$$\|(\mathcal{H}_{ac}^{s/2} - \mathcal{H}_0^{s/2})f\|_{L^p(\mathbb{R})} \leq C\|f\|_{L^q(\mathbb{R})},$$

*for  $1/p - 1/q = s$  and  $f \in S(\mathbb{R})$ .*

It is natural to use a Paley-Littlewood localization associated with the perturbed Hamiltonian. Here and below  $\varphi(\tau) \in C_0^\infty(\mathbb{R} \setminus 0)$  is a non-negative even function, such that

$$\sum_{j \in \mathbb{Z}} \varphi\left(\frac{\tau}{2^j}\right) = 1, \quad \forall \tau \in \mathbb{R} \setminus 0 \quad (2.6)$$

and

$$\varphi\left(\frac{\tau}{2^k}\right) \varphi\left(\frac{\tau}{2^\ell}\right) = 0, \quad \forall k, \ell \in \mathbb{Z}, \quad |k - \ell| \geq 2. \quad (2.7)$$

We set

$$\pi_k^{ac} = \varphi\left(\frac{\sqrt{\mathcal{H}_{ac}}}{2^k}\right), \quad \pi_k^0 = \varphi\left(\frac{\sqrt{\mathcal{H}_0}}{2^k}\right). \quad (2.8)$$

We have the following equivalent norm (see [21])

$$\|f\|_{\dot{H}_{p, \mathcal{H}_{ac}}^s(\mathbb{R})} \sim \left\| \left( \sum_{k=-\infty}^{\infty} 2^{2ks} |\pi_k^{ac} f|^2 \right)^{1/2} \right\|_{L^p(\mathbb{R})}. \quad (2.9)$$

Our approach to prove Theorem 2 is based on establishing estimate of the type.

**Lemma 1.** *If the assumptions of Theorem 2 are fulfilled, then for any  $s \in (0, 1/p)$  and  $q \in (1, \infty)$  defined by*

$$\frac{1}{p} - \frac{1}{q} = s$$

*we have*

$$\left\| 2^{ks} (\pi_k^{ac} - \pi_k^0) f \right\|_{\ell_k^2} \Big\|_{L_x^p(\mathbb{R})} \leq C\|f\|_{L^q(\mathbb{R})}. \quad (2.10)$$

Indeed if this estimate is verified, then we can use (2.9) and see that (2.10) implies the assertion of Theorem 2.

Therefore, the estimate (2.10) is the key point in the proof of Theorem 2.

**Corollary 1.** *If the assumptions of Theorem 2 are fulfilled, then the equivalence property (1.3) holds.*

*Proof.* The results in [4], [17], [1], [3], [21] imply the existence and continuity of the wave operators in  $L^p$ ,  $1 < p < \infty$ , so one can deduce Bernstein inequality

$$\|\pi_k^{ac} f\|_{L^q(\mathbb{R})} \leq C(2^k)^{1/p-1/q} \|f\|_{L^p(\mathbb{R})}, \quad 1 \leq p \leq q \leq \infty, \quad k \in \mathbb{Z} \quad (2.11)$$

and via the equivalence property (2.9) we deduce the Sobolev estimate

$$\|f\|_{L^q(\mathbb{R})} \leq C\|\mathcal{H}_{ac}^{s/2} f\|_{L^p(\mathbb{R})}, \quad 1 < p < q < \infty, \quad s = \frac{1}{p} - \frac{1}{q}. \quad (2.12)$$

From the estimate of Theorem 2 now we can write

$$\|(\mathcal{H}_{ac}^{s/2} - \mathcal{H}_0^{s/2})f\|_{L^p(\mathbb{R})} \leq C\|f\|_{L^q(\mathbb{R})} \leq C\|\mathcal{H}_{ac}^{s/2} f\|_{L^p(\mathbb{R})},$$

so we have

$$\|\mathcal{H}_0^{s/2} f\|_{L^p(\mathbb{R})} \leq C\|\mathcal{H}_{ac}^{s/2} f\|_{L^p(\mathbb{R})}.$$

The opposite estimate can be deduced in the same way from Theorem 2 and the "free" Sobolev estimate

$$\|f\|_{L^q(\mathbb{R})} \leq C \|\mathcal{H}_0^{s/2} f\|_{L^p(\mathbb{R})}, \quad 1 < p < q < \infty, \quad s = \frac{1}{p} - \frac{1}{q}. \quad (2.13)$$

This completes the proof.  $\square$

Theorem 2 has also the following simple consequences.

**Corollary 2.** *If the assumptions of Theorem 2 are fulfilled, then the Hardy inequality (1.2) holds.*

**Corollary 3.** *If the assumptions of Theorem 2 are fulfilled, then we have the fractional Leibnitz rule, i.e.*

$$\|\mathcal{H}_{ac}^{s/2}(fg) - f\mathcal{H}_{ac}^{s/2}g - g\mathcal{H}_{ac}^{s/2}f\|_{L^p(\mathbb{R})} \leq C \|\mathcal{H}_{ac}^{s_1/2}f\|_{L^{p_1}(\mathbb{R})} \|\mathcal{H}_{ac}^{s_2/2}g\|_{L^{p_2}(\mathbb{R})}, \quad (2.14)$$

provided

$$1 < p, p_1, p_2 < \infty, \quad \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} \quad (2.15)$$

and

$$0 < s = s_1 + s_2, \quad s_1, s_2 \geq 0, \quad s_1 < \frac{1}{p_1}, \quad s_2 < \frac{1}{p_2}.$$

Alternative application of the equivalence of the homogeneous Sobolev norms can be connected with the fractional power of the pseudo conformal generators, defined by

$$|J_0(t)|^s = t e^{ix^2/(4t)} \mathcal{H}_0^{s/2} e^{-ix^2/(4t)}, \quad s \geq 0. \quad (2.16)$$

These operators commute with the free Schrödinger group  $e^{-i\mathcal{H}_0 t}$ ,  $\mathcal{H}_0 = -\partial_x^2$ .

Natural generalization of (2.16) for the case of perturbed Schrödinger group  $e^{-i\mathcal{H}t}$ ,  $\mathcal{H} = -\partial_x^2 + V$  with short range potential is introduced in [2] as follows

$$|J(t)|^s = t e^{ix^2/(4t)} \mathcal{H}^{s/2} e^{-ix^2/(4t)}, \quad s \geq 0. \quad (2.17)$$

In the case  $V = 0$  we have

$$\|(t\partial_x - ix/2)(e^{-i\mathcal{H}_0 t} f)\|_{L^2} \sim \| |J_0(t)|^s (e^{-i\mathcal{H}_0 t} f) \|_{L^2}$$

so the conservation of the pseudo conformal energy

$$\|(t\partial_x - ix/2)(e^{-i\mathcal{H}_0 t} f)\|_{L^2} = \frac{1}{2} \|xf\|_{L^2} \quad (2.18)$$

and interpolation argument imply

$$\| |J_0(t)|^s (e^{-i\mathcal{H}_0 t} f) \|_{L^2} \leq C (\|f\|_{H^1(\mathbb{R})} + \|xf\|_{L^2}), \quad \forall t > 0, \quad (2.19)$$

for any  $s \in [0, 1]$ .

One can use the the equivalence result as stated in Theorem 2 and deduce (see Lemma 5.1 in [2])

$$\| |J_0(t)|^s g \|_{L^2} \sim \| |J(t)|^s g \|_{L^2} \quad (2.20)$$

for any  $s \in [0, 1/2]$ .

We turn now to possible inflation phenomena manifested by the pseudo conformal norms over the perturbed Schrödinger flow, i.e. we shall study the quantity

$$\| |J_0(t)|^s (e^{-i\mathcal{H}t} f) \|_{L^2}$$

when  $s = 1 > 1/2$ .

**Lemma 2.** *Assume the potential  $V \in L^\infty \cap L_\gamma^1(\mathbb{R})$  with  $\gamma > 1$  is such that*

$$\int_{\mathbb{R}} V(y) dy > 0. \quad (2.21)$$

*Then for any initial data  $f(x) \in S(\mathbb{R})$  with*

$$f(0) \neq 0 \quad (2.22)$$

*we have*

$$\limsup_{t \rightarrow \infty} \|(t\partial_x - ix/2)(e^{-i\mathcal{H}t} f)\|_{L^2} = \infty. \quad (2.23)$$

### 3 Idea to prove the key Lemma 1

Our main tool to study the kernel

$$\varphi\left(\frac{\sqrt{\mathcal{H}_{ac}}}{M}\right)(x, y)$$

is the following representation of the kernel as filtered Fourier transform

$$\mathcal{F}_{\varphi, M}(a)(\xi) = \int \varphi\left(\frac{\tau}{M}\right) a(\tau) e^{-i\xi\tau} d\tau \quad (3.1)$$

of symbols  $a(\tau)$  represented as linear combinations with constant coefficients of functions in the set

$$\mathcal{A} = \{ 1, T(\tau), R_{\pm}(\tau) \}, \quad (3.2)$$

or more generally of symbols involving functions  $a(x, \tau)$  represented as linear combinations with constant coefficients of functions in the set

$$\mathcal{B} = \{ \widetilde{m_{\pm}}(x, \tau), T(\tau)\widetilde{m_{\pm}}(x, \tau), R_{\pm}(\tau)\widetilde{m_{\pm}}(x, \tau) \}, \quad (3.3)$$

where  $\widetilde{m_{\pm}}(x, \tau) = m_{\pm}(x, \tau) - 1$ ,  $m_{\pm}$  are modified Jost functions, while  $T, R_{\pm}$  are the transmission and reflection coefficients.

It is simple to establish that the kernel  $\varphi(\sqrt{\mathcal{H}}/M)(x, y)$  can be decomposed as follows (one can see [6]):

**Lemma 3.1.** *If  $\varphi$  is an even non-negative function, such that  $\varphi \in C_0^{\infty}(\mathbf{R} \setminus \{0\})$ , then for any  $M > 0$  we have*

$$\varphi\left(\frac{\sqrt{\mathcal{H}}}{M}\right)(x, y) = K_M^0(x, y) + \widetilde{K}_M(x, y), \quad (3.4)$$

where  $K_M^0(x, y)$  can be represented as sum of the terms

$$\mathbb{1}_{\epsilon_1 x > 0} \mathbb{1}_{\epsilon_2 y > 0} \mathcal{F}_{\varphi, M}(a)(\epsilon_3 x + \epsilon_4 y) \quad (3.5)$$

and the term  $\widetilde{K}_M(x, y)$  is represented as sum of the terms

$$\begin{aligned} & \mathbb{1}_{\epsilon_1 x > 0} \mathbb{1}_{\epsilon_2 y > 0} \mathcal{F}_{\varphi, M}(b_1(x, \cdot))(\epsilon_3 x + \epsilon_4 y) + \mathbb{1}_{\epsilon_1 x > 0} \mathbb{1}_{\epsilon_2 y > 0} \mathcal{F}_{\varphi, M}(b_2(y, \cdot))(\epsilon_3 x + \epsilon_4 y) + \\ & + \mathbb{1}_{\epsilon_1 x > 0} \mathbb{1}_{\epsilon_2 y > 0} \mathcal{F}_{\varphi, M}(b_3(x, \cdot) b_4(y, \cdot))(\epsilon_3 x + \epsilon_4 y), \end{aligned} \quad (3.6)$$

where  $\epsilon_i = \pm 1$ , for  $i = 1, \dots, 4$ ,  $a(\tau)$  represents a linear combination with constant coefficients of functions in the set  $\mathcal{A}$  in (3.2) and  $b_i$ , for  $i = 1, \dots, 4$ , are linear combinations with constant coefficients of functions in the set  $\mathcal{B}$  in (3.3).

*Remark 3.2.* We shall call the term  $K_M^0(x, y)$  the leading one, with the following exact representation

$$K_M^0(x, y) = c \int_{\mathbf{R}} e^{-i\tau(x-y)} \varphi\left(\frac{\tau}{M}\right) \alpha(x, y, \tau) d\tau \quad (3.7)$$

with symmetric kernel  $\alpha(x, y, \tau) = \alpha(y, x, \tau)$  and

$$\alpha(x, y, \tau) = \begin{cases} T(\tau) & x < 0 < y, \\ (R_+(\tau) + 1)e^{2i\tau x} - e^{2i\tau y} + 1 & 0 < x < y, \\ (R_-(\tau) + 1)e^{-2i\tau y} - e^{-2i\tau x} + 1 & x < y < 0. \end{cases}$$

The term  $\widetilde{K}_M(x, y)$  will be called the remainder one. In Lemma 3.1 to simplify the notation we neglected the symbolism  $a^{\pm}, b_i^{\pm}$ .

A priori estimates for the remainder term are obtained using the estimates of the filtered Fourier transform established in Lemma 5.4 and Lemma 5.5.

**Lemma 3.3.** Suppose the condition (2.4) is fulfilled with  $\gamma \geq 1 + s$ ,  $s \in (0, 1)$ , the operator  $\mathcal{H}$  has no point spectrum and 0 is not a resonance point for  $\mathcal{H}$ . If  $\varphi$  is an even non-negative function, such that  $\varphi \in C_0^\infty(\mathbf{R} \setminus \{0\})$ , then for any  $p \in (1, 1/s)$ , any  $M \in (0, \infty)$  and for any  $b^\pm(x, \tau)$ ,  $b_1^\pm(x, \tau)$ ,  $b_2^\pm(x, \tau)$  in the set (3.3) we have

$$\begin{aligned} & \left\| \int_{\mathbf{R}} \mathbf{1}_{\pm x > 0} \mathcal{F}_{\varphi, M}(b^\pm(x, \cdot))(x \pm y) f(y) dy \right\|_{L_x^p(\mathbf{R})} + \\ & + \left\| \int_{\mathbf{R}} \mathbf{1}_{\pm y > 0} \mathcal{F}_{\varphi, M}(b^\pm(y, \cdot))(x \pm y) f(y) dy \right\|_{L_x^p(\mathbf{R})} \leq \frac{C}{\langle M \rangle} \|f\|_{L^q(\mathbf{R})}, \end{aligned} \quad (3.8)$$

and

$$\left\| \int_{\mathbf{R}} \mathbf{1}_{\pm x > 0} \mathbf{1}_{\pm y > 0} \mathcal{F}_{\varphi, M}(b_1^\pm(x, \cdot) b_2^\pm(y, \cdot))(x \pm y) f(y) dy \right\|_{L_x^p(\mathbf{R})} \leq \frac{C}{\langle M \rangle} \|f\|_{L^q(\mathbf{R})}, \quad (3.9)$$

where  $\frac{1}{q} = \frac{1}{p} - s$ .

According with the notation introduced in (2.8), we set

$$\pi_{\leq k}^{ac} = \sum_{j \leq k} \pi_j^{ac}, \quad \pi_{\geq k}^{ac} = \sum_{j \geq k} \pi_j^{ac}. \quad (3.10)$$

$$f_k = \pi_k^{ac} f, \quad f_{\leq k} = \sum_{j \leq k} \pi_j^{ac} f, \quad f_{\geq k} = \sum_{j \geq k} \pi_j^{ac} f, \quad f_{k_1, k_2} = \sum_{k_1 \leq j \leq k_2} \pi_j^{ac} f$$

and respectively  $f_k^0$ ,  $f_{\leq k}^0$ ,  $f_{\geq k}^0$ ,  $f_{k_1, k_2}^0$  defined as before replacing  $\pi_j^{ac}$  with  $\pi_j^0$ .

Hence, the decomposition (3.4) can be rewritten as follows

$$\pi_k^{ac} = I_k - (\pi_k^{ac} - I_k),$$

where the operator  $I_k$  represents the operators involved in the leading kernel and  $(\pi_k^{ac} - I_k)$  is the remainder term.

To prove Lemma 1 we will establish the following inequalities:

$$\left\| \left\| 2^{ks} (\pi_k^{ac} - I_k) f \right\|_{\ell_k^2} \right\|_{L_x^p(\mathbf{R})} \leq C \|f\|_{L^q(\mathbf{R})}, \quad (3.11)$$

$$\left\| \left\| 2^{ks} (I_k - \pi_k^0) f \right\|_{\ell_k^2} \right\|_{L_x^p(\mathbf{R})} \leq C \|f\|_{L^q(\mathbf{R})}, \quad (3.12)$$

with  $1/p = 1/q + s$  and  $I_k$  are the operators

$$I_k(f)(x) = \int_{\mathbf{R}} K_{2k}^0(x, y) f(y) dy$$

with kernels representing the leading term (3.5) in the expansion of Lemma 3.1 of  $\pi_k$ .

## 4 Sup and Hölder type arproi estimates

### 4.1 Estimates for the modified Jost functions

In this section we recall some classical results concerning the spectral decomposition of the perturbed Hamiltonian. Recall that the Jost functions are solutions  $f_\pm(x, \tau) = e^{\pm i\tau x} m_\pm(x, \tau)$  of  $\mathcal{H}u = \tau^2 u$  with

$$\lim_{x \rightarrow +\infty} m_+(x, \tau) = 1 = \lim_{x \rightarrow -\infty} m_-(x, \tau).$$

We set  $x_+ := \max\{0, x\}$ ,  $x_- := \max\{0, -x\}$ .

The estimate and the asymptotic expansions of  $m_\pm(x, \tau)$  are based on the following integral equations

$$m_\pm(x, \tau) = 1 + K_\pm^{(\tau)}(m_\pm(\cdot, \tau))(x), \quad (4.1)$$

where  $K_{\pm}^{(\tau)}$  is the integral operator defined as follows

$$K_{\pm}^{(\tau)}(f)(x) = \pm \int_x^{\pm\infty} D(\pm(t-x), \tau) V(t) f(t) dt$$

and

$$D(t, \tau) = \frac{e^{2it\tau} - 1}{2i\tau} = \int_0^t e^{2iy\tau} dy; \quad (4.2)$$

The following lemma is well known.

**Lemma 4.1.** (see Lemma 1 p. 130 [4] and Lemma 2.1 in [18]) Assume  $V \in L_{\gamma}^1(\mathbb{R})$ ,  $\gamma \in (1, 2]$ . Then we have the properties:

a) for any  $x \in \mathbb{R}$  the function

$$\tau \in \overline{\mathbb{C}_{\pm}} \mapsto m_{\pm}(x, \tau), \quad \mathbb{C}_{\pm} = \{\tau \in \mathbb{C}; \operatorname{Im} \tau \gtrless 0\} \quad (4.3)$$

is analytic in  $\mathbb{C}_{\pm}$  and  $C^1(\overline{\mathbb{C}_{\pm}})$ ;

b) there exist constants  $C_1$  and  $C_2 > 0$  such that for any  $x, \tau \in \mathbb{R}$ :

$$\mathbf{1}_{\pm x > 0} |m_{\pm}(x, \tau) - 1| \leq C_1 \langle \tau \rangle^{-1}; \quad (4.4)$$

$$\mathbf{1}_{\pm x > 0} |\partial_{\tau} m_{\pm}(x, \tau)| \leq \frac{C_2}{|\tau|^{\gamma-2} \langle \tau \rangle^{\gamma-1}}. \quad (4.5)$$

A slight improvement is given in the next Lemma.

**Lemma 4.2.** ( see [6]) Suppose  $V \in L_{\gamma}^1(\mathbb{R})$  with  $\gamma \geq 1$ . Then we have the following properties:

a) There exists a constant  $C > 0$  such that for any  $x \in \mathbb{R}$ ,  $\tau \in \overline{\mathbb{C}_{\pm}}$ , we have

$$|m_{\pm}(x, \tau) - 1| \leq C \frac{\langle x_{\mp} \rangle}{\langle x_{\pm} \rangle^{\gamma-1}}; \quad (4.6)$$

b) There exists a constant  $C > 0$  such that for any  $x \in \mathbb{R}$ ,  $\tau \in \overline{\mathbb{C}_{\pm}} \setminus \{0\}$ , we have

$$|m_{\pm}(x, \tau) - 1| \leq C \frac{\langle x_{\mp} \rangle}{\langle x_{\pm} \rangle^{\gamma} |\tau|}; \quad (4.7)$$

c) Let  $\sigma \in [0, 1)$ . Then there exists a constant  $C > 0$  such that for any  $x \in \mathbb{R}$  we have

$$\|m_{\pm}(x, \tau) - 1\|_{C^{0,\sigma}(\mathbb{C}_{\pm})} \leq C \frac{\langle x_{\mp} \rangle^{1+\sigma}}{\langle x_{\pm} \rangle^{\gamma-1-\sigma}}, \quad \gamma > 1, \quad 0 \leq \sigma \leq \gamma - 1; \quad (4.8)$$

d) Let  $\sigma \in [0, 1)$ . Then there exists a constant  $C > 0$  such that for any  $x \in \mathbb{R}$  we have

$$\|\tau(m_{\pm}(x, \tau) - 1)\|_{C^{0,\sigma}(\mathbb{C}_{\pm})} \leq C \frac{\langle x_{\mp} \rangle^{1+\sigma}}{\langle x_{\pm} \rangle^{\gamma-\sigma}}, \quad \gamma > 1. \quad (4.9)$$

## 4.2 Estimates for transmission and reflection coefficients

The transmission coefficient  $T(\tau)$  and the reflection coefficients  $R_{\pm}(\tau)$  are defined by the formula

$$T(\tau) m_{\mp}(x, \tau) = R_{\pm}(\tau) e^{\pm 2i\tau x} m_{\pm}(x, \tau) + m_{\pm}(x, -\tau). \quad (4.10)$$

From [4] and from [18] we have the following lemma.

**Lemma 4.3.** We have the following properties of the transmissions and reflection coefficients.



a)  $T, R_{\pm} \in C(\mathbb{R})$ .

b) There exists  $C_1, C_2 > 0$  such that:

$$|T(\tau) - 1| + |R_{\pm}(\tau)| \leq C_1 \langle \tau \rangle^{-1} \quad (4.11)$$

$$|T(\tau)|^2 + |R_{\pm}(\tau)|^2 = 1. \quad (4.12)$$

c) If  $T(0) = 0$ , (i.e. zero is not a resonance point), then for some  $\alpha \in \mathbb{C} \setminus \{0\}$  and for some  $\alpha_+, \alpha_- \in \mathbb{C}$

$$\begin{aligned} T(\tau) &= \alpha\tau + o(\tau), \quad 1 + R_{\pm}(\tau) = \alpha_{\pm}\tau + o(\tau) \quad \text{as } \tau \rightarrow 0, \\ T(\tau) &= 1 + O(|\tau|^{-1}), \quad R_{\pm}(\tau) = O(|\tau|^{-1}) \quad \text{as } \tau \rightarrow \infty. \end{aligned} \quad (4.13)$$

d) there exists a constant  $C > 0$  such that for any  $\tau \in \mathbb{R}$ :

$$T'(\tau) \leq C \langle \tau \rangle^{-1}. \quad (4.14)$$

The property c) in the last Lemma suggests the following.

**Definition 4.4.** The origin is a resonance point for the hamiltonian  $\mathcal{H}$  if and only if

$$T(0) \neq 0.$$

Therefore, taking a bump function  $\varphi \in C_0^\infty((0, \infty))$  (with support in  $[1/2, 2]$  for example), we have estimates in the algebra  $C([0, 4])$  of the terms of type

$$\|\varphi(\cdot)T(M\cdot)\|_{C^0([0,4])} + \|\varphi(\cdot)(R_{\pm}(M\cdot) + 1)\|_{C^0([0,4])} \leq CM \quad (4.15)$$

and

$$\left\| \frac{\varphi(\cdot)}{T(M\cdot)} \right\|_{C^0([0,4])} + \left\| \frac{\varphi(\cdot)}{(R_{\pm}(M\cdot) + 1)} \right\|_{C^0([0,4])} \leq CM^{-1} \quad (4.16)$$

for  $M \in (0, 1]$ .

We can use the assumption  $V \in L_\gamma^1(\mathbb{R})$ ,  $\gamma > 1$ , to get some more precise Hölder type bounds.

**Lemma 4.5.** Suppose  $V \in L_\gamma^1(\mathbb{R})$  with  $\gamma > 1$  and  $T(0) = 0$ . Then for any  $\sigma \in (0, s]$  and  $M \in (0, 1]$  we have:

a)  $T, R_{\pm} \in C^{0,\sigma}(\mathbb{R})$ ;

b) for  $M \in (0, 1)$  we have

$$\|\varphi(\cdot)T(M\cdot)\|_{C^{0,\sigma}((0,+\infty))} + \|\varphi(\cdot)(R_{\pm}(M\cdot) + 1)\|_{C^{0,\sigma}((1/2,2))} \leq CM; \quad (4.17)$$

c) for  $M \in [1, \infty)$  we have

$$\|\varphi(\cdot)(T(M\cdot) - 1)\|_{C^{0,\sigma}((0,+\infty))} + \|\varphi(\cdot)R_{\pm}(M\cdot)\|_{C^{0,\sigma}((1/2,2))} \leq CM^{-1}. \quad (4.18)$$

*Proof.* The proof is based on the relations

$$\frac{\tau}{T(\tau)} = \tau - \frac{1}{2i} \int_{\mathbb{R}} V(t)m_+(t, \tau)dt, \quad \tau \in \mathbb{R} \setminus \{0\}, \quad (4.19)$$

$$R_{\pm}(\tau) = \frac{T(\tau)}{2i\tau} \int_{\mathbb{R}} e^{\mp 2it\tau} V(t)m_{\mp}(t, \tau)dt, \quad \tau \in \mathbb{R} \setminus \{0\} \quad (4.20)$$

and the properties of the functions  $m_{\mp}(t, \tau)$  from Lemma 4.2. Indeed, we can get the estimates

$$\left\| \frac{\varphi(\cdot)}{T(M\cdot)} \right\|_{C^{0,\sigma}([0,4])} + \left\| \frac{\varphi(\cdot)}{(R_{\pm}(M\cdot) + 1)} \right\|_{C^{0,\sigma}([0,4])} \leq CM^{-1} \quad (4.21)$$

first. Further, we can use the fact<sup>2</sup> that we can control the norm of the inverse of  $f$  in the subalgebra  $C^{0,\sigma}$  by the norm of  $f$  in  $C^{0,\sigma}$  and the norm of  $1/f$  in  $C(T)$

$$\left\| \frac{\varphi(\cdot)}{f(\cdot)} \right\|_{C^{0,\sigma}([0,4])} \leq C \left\| \frac{\tilde{\varphi}(\cdot)}{f(\cdot)} \right\|_{C^0([0,4])} + \frac{\|\tilde{\varphi}(\cdot)f\|_{C^{0,\sigma}([0,4])}}{\|f(\cdot)\|_{C^0([0,4])}^2},$$

where  $\tilde{\varphi} \in C_0^\infty((0, \infty))$  has slightly larger support in  $[1/2 - \delta, 2 + \delta]$  with  $\delta > 0$  sufficiently small. Applying this estimate and the estimate (4.16) and (4.21) with  $\varphi$  replaced by a cut-off function with slightly larger support, we complete the proof.  $\square$

## 5 Estimates of the filtered Fourier transform of $m_\pm - 1$

Given a bump function  $\varphi \in C_0^\infty(\mathbb{R})$ , we define the corresponding filtered Fourier transform as in (3.1). We shall distinguish two different cases. If the bump function  $\varphi \in C_0^\infty((0, \infty))$  is such that (2.6) and (2.7) are satisfied, then we can assert that  $\varphi(\tau/M)$  has a support with  $\tau \sim M$ .

The integral equation (4.1) with sign  $+$  can be rewritten as

$$\widetilde{m}_+(x, \tau) = \int_x^\infty \int_0^{t-x} e^{2i\tau y} V(t) dy dt + \int_x^\infty \int_0^{t-x} e^{2i\tau y} V(t) \widetilde{m}_+(t, \tau) dy dt, \quad (5.1)$$

where

$$\widetilde{m}_+(x, \tau) = m_+(x, \tau) - 1.$$

If we assume that  $V \in L_\gamma^1(\mathbb{R})$ ,  $\gamma = 1 + s$ , then the assertion of Lemma 4.2 guarantees that  $\widetilde{m}_+(x, \tau) = m_+(x, \tau) - 1$  is in  $L_{x>0}^1(\mathbb{R})$ .

Applying the filtered Fourier transform and setting

$$g_M(\xi; x) = \int_{\mathbb{R}} e^{-i\tau\xi} \widetilde{m}_+(x, \tau) \varphi\left(\frac{\tau}{M}\right) d\tau = \mathcal{F}_{\varphi, M}(\widetilde{m}_+(x, \cdot))(\xi),$$

we get

$$\begin{aligned} g_M(\xi; x) &= M \underbrace{\int_x^\infty \int_0^{t-x} V(t) \widehat{\varphi}(M(\xi - 2y)) dy dt}_{a_M(\xi; x)} \\ &\quad + \int_x^\infty \int_0^{t-x} V(t) g_M(\xi - 2y; t) dy dt. \end{aligned} \quad (5.2)$$

We have the following pointwise estimates.

**Lemma 5.1.** *If  $\varphi \in C_0^\infty(\mathbb{R})$ , satisfies (2.6), (2.7) and  $V \in L_\gamma^1(\mathbb{R})$ ,  $\gamma = 1 + s$ ,  $s \in (0, 1)$ , then for  $M \in (0, 1)$  the filtered Fourier transform*

$$\mathcal{F}_{\varphi, M}(\widetilde{m}_\pm(x, \cdot))(\xi) = \int_{\mathbb{R}} e^{-i\tau\xi} \widetilde{m}_\pm(x, \tau) \varphi\left(\frac{\tau}{M}\right) d\tau$$

*satisfies the pointwise estimates:*

- one can find functions

$$F_M^\pm(\xi) \in L^1(\mathbb{R}), \quad \|F_M^\pm\|_{L^1(\mathbb{R})} \leq C(\|V\|_{L_{1+s}^1(\mathbb{R})}) \|\widehat{\varphi}\|_{L^1(\mathbb{R})},$$

*so that*

$$\mathbb{1}_{\{\pm x > 0\}} \langle x \rangle^s |\mathcal{F}_{\varphi, M}(\widetilde{m}_\pm(x, \cdot))(\xi)| \leq F_M^\pm(\xi). \quad (5.3)$$

---

<sup>2</sup>the problem to have norm-controlled inversion in smooth Banach algebra is well-known and some more general results and references can be found in [10]

*Proof.* We choose the sign  $+$  in (5.3) for determinacy. To prove (5.3) we set

$$G_M(\xi; x) = \mathbf{1}_{\{x>0\}} \sup_{\eta < \xi} |g_M(\eta; x)| \langle x \rangle^s,$$

where  $g_M(\xi; x)$  is the Filtered Fourier transform of the remainder  $\widetilde{m}_+(x, \tau) = m_+(x, \tau) - 1$ , satisfying the integral equation (5.2). The function

$$F_M(\xi) = M \int_0^\infty \langle t \rangle^\gamma |V(t)| \int_0^t |\widehat{\varphi}(M(\xi - 2y))| dy dt, \quad (5.4)$$

satisfies

$$F_M(\xi) \in L^1(\mathbb{R}), \quad \|F_M\|_{L^1(\mathbb{R})} \leq \|V\|_{L^\gamma_1(\mathbb{R})} \|\widehat{\varphi}\|_{L^1(\mathbb{R})}. \quad (5.5)$$

Moreover, since we are considering the case  $x > 0$  we get easily the following estimates

$$|\mathbf{1}_{x>0} \langle x \rangle^s a_M(\xi; x)| \leq F_M(\xi),$$

where  $a_M(\xi; x)$  is defined in (5.2). Hence, coming back to  $G_M(\xi; x)$  and recalling (5.2) we have

$$G_M(\xi; x) \leq F_M(\xi) + \int_x^\infty \langle t \rangle |V(t)| G_M(\xi; t) dt, \quad \forall x > 0. \quad (5.6)$$

Applying the Gronwall lemma we get

$$G_M(\xi; x) \leq C F_M(\xi),$$

where  $C$  is a positive constant depending on  $\|V\|_{L^1_1(\mathbb{R})}$  and  $F_M(\xi)$  satisfies (5.4) and (5.5). This completes the proof.  $\square$

If  $M \geq 1$  and  $\varphi$  satisfying (2.6) and (2.7), then we can improve the results of Lemma 5.1. Indeed, the term  $a_M(\xi; x)$  in (5.2) can be rewritten as follows

$$a_M(\xi; x) = M \int_x^\infty dt \int_{\mathbb{R}} d\tau V(t) e^{-i\tau M\xi} \varphi(\tau) \frac{e^{2iM\tau(x-y)} - 1}{2iM\tau}.$$

Hence we have that

$$|\mathbf{1}_{x>0} \langle x \rangle^s a_M(\xi; x)| \leq F_M^{(1)}(\xi),$$

where

$$F_M^{(1)}(\xi) = \int_x^\infty \langle t \rangle^s |V(t)| |\widehat{\varphi}(M\xi)| dt \quad (5.7)$$

and

$$\|F_M^{(1)}(\xi)\|_{L^1(\mathbb{R})} \leq \frac{1}{M} \|V\|_{L^1_s(\mathbb{R})} \|\widehat{\varphi}\|_{L^1(\mathbb{R})}.$$

Proceeding as in the proof of Lemma 5.1 we get the following result.

**Lemma 5.2.** *If  $\varphi$  satisfies (2.6) and (2.7) and  $V \in L^\gamma_1(\mathbb{R})$ ,  $\gamma = 1 + s$ ,  $s \in (0, 1)$ , then for  $M \in (0, \infty)$  the filtered Fourier transform*

$$\mathcal{F}_{\varphi, M}(\widetilde{m}_\pm(x, \cdot))(\xi) = \int_{\mathbb{R}} e^{-i\tau\xi} (\widetilde{m}_\pm(x, \tau)) \varphi\left(\frac{\tau}{M}\right) d\tau$$

*satisfies the pointwise estimates:*

- one can find functions

$$F_M^\pm(\xi) \in L^1(\mathbb{R}), \quad \|F_M^\pm\|_{L^1(\mathbb{R})} \leq \frac{1}{\langle M \rangle} C(\|V\|_{L^{1+s}_1(\mathbb{R})}) \|\widehat{\varphi}\|_{L^1(\mathbb{R})},$$

*so that*

$$\mathbf{1}_{\{\pm x > 0\}} \langle x \rangle^s |\mathcal{F}_{\varphi, M}(\widetilde{m}_\pm(x, \cdot))(\xi)| \leq F_M^\pm(\xi). \quad (5.8)$$

One can use a Wiener type argument and deduce estimates for  $T(\tau), R_{\pm}(\tau) + 1$ .

**Lemma 5.3.** (see [3], [21]) *If  $\varphi \in C_0^\infty(\mathbb{R})$  obeys (2.6), (2.7) and  $V \in L_\gamma^1(\mathbb{R})$ ,  $\gamma = 1 + s$ ,  $s \in (0, 1)$ , then for  $M \in (0, \infty)$  the filtered Fourier transforms*

$$\mathcal{F}_{\varphi, M}(T(\cdot))(\xi) = \int_{\mathbb{R}} e^{-i\tau\xi} T(\tau) \varphi\left(\frac{\tau}{M}\right) d\tau$$

and

$$\mathcal{F}_{\varphi, M}(R_{\pm}(\cdot) + 1)(\xi) = \int_{\mathbb{R}} e^{-i\tau\xi} (R_{\pm}(\tau) + 1) \varphi\left(\frac{\tau}{M}\right) d\tau$$

are in  $L^1(\mathbb{R})$  and the following inequality are satisfied

$$\begin{aligned} \|\mathcal{F}_{\varphi, M}(T(\cdot))(\xi)\|_{L^1(\mathbb{R})} + \|\mathcal{F}_{\varphi, M}(R_{\pm}(\cdot) + 1)(\xi)\|_{L^1(\mathbb{R})} &\leq C(\|V\|_{L_{1+s}^1(\mathbb{R})}) \|\widehat{\varphi}\|_{L^1(\mathbb{R})}, \quad M \in (0, 1), \\ \|\mathcal{F}_{\varphi, M}(T(\cdot) - 1)(\xi)\|_{L^1(\mathbb{R})} + \|\mathcal{F}_{\varphi, M}R_{\pm}(\cdot)(\xi)\|_{L^1(\mathbb{R})} &\leq \frac{1}{\langle M \rangle} C(\|V\|_{L_{1+s}^1(\mathbb{R})}) \|\widehat{\varphi}\|_{L^1(\mathbb{R})}, \quad M > 1. \end{aligned}$$

Turning to the estimates (5.3), we see that

$$a(x, \xi) = \mathbb{1}_{\{\pm x > 0\}} \mathcal{F}_{\varphi, M}(\widetilde{m_{\pm}(x, \cdot)})(\xi)$$

satisfies estimate

$$|a(x, \xi)| \leq a_1(x) a_2(\xi), \quad a_1 \in L^{1/s, \infty}(\mathbb{R}), \quad a_2 \in L^1(\mathbb{R}), \quad (5.9)$$

where  $a_1(x) = \langle x \rangle^{-s}$ . Lemma 5.3 guarantees that

$$b(\xi) = \mathcal{F}_{\varphi, M}(T(\cdot))(\xi) \in L^1(\mathbb{R}).$$

Since

$$\mathbb{1}_{\{\pm x > 0\}} \mathcal{F}_{\varphi, M}(T(\cdot)(\widetilde{m_{\pm}(x, \cdot)}))(\xi) = a(x, \cdot) * b(\cdot)(\xi),$$

we see that

$$|a(x, \cdot) * b(\cdot)(\xi)| \leq a_1(x) \underbrace{a_2 * |b|}_{\widetilde{a_2}}(\xi), \quad a_1 \in L^{1/s, \infty}(\mathbb{R}), \quad \widetilde{a_2} \in L^1(\mathbb{R}),$$

since

$$L^1 * L^1 \subset L^1$$

due to the Young inequality.

The above inclusion actually can be modified in a way suitable for our a priori estimates as follows

$$(L^1 \cap L^\infty) * (L^1 \cap L^\infty) \subset (L^1 \cap L^\infty). \quad (5.10)$$

This observation leads to the following.

**Lemma 5.4.** *If  $\varphi \in C_0^\infty(\mathbb{R})$ ,  $V \in L_\gamma^1(\mathbb{R})$ ,  $\gamma = 1 + s$ ,  $s \in (0, 1)$ , and  $a^\pm(x, \tau)$  is any function in the set*

$$\{\widetilde{m_{\pm}(x, \tau)}, \quad T(\tau)\widetilde{m_{\pm}(x, \tau)}, \quad (R_{\pm}(\tau) + 1)\widetilde{m_{\pm}(x, \tau)}\}, \quad (5.11)$$

then for  $M \in (0, \infty)$  the filtered Fourier transform

$$\mathcal{F}_{\varphi, M}(a^\pm(x, \cdot))(\xi) = \int_{\mathbb{R}} e^{-i\tau\xi} a^\pm(x, \tau) \varphi\left(\frac{\tau}{M}\right) d\tau$$

satisfies the pointwise estimates:

$$\mathbb{1}_{\{\pm x > 0\}} |\mathcal{F}_{\varphi, M}(a^\pm(x, \cdot))(\xi)| \leq f_1(x) f_2^{(M)}(\xi), \quad (5.12)$$

where

$$f_1(x) \in L^{1/s, \infty}(\mathbb{R}) \cap L^\infty(\mathbb{R}), \quad f_2^{(M)}(\xi) \in L^1(\mathbb{R})$$

and  $\|f_2^{(M)}\|_{L^1(\mathbb{R})} \leq C/\langle M \rangle$ .

Finally we consider products of type  $a^\pm(x, \tau)b^\pm(y, \tau)$ , where  $a, b$  are in the set (5.11) and we have the following estimates.

**Lemma 5.5.** *If  $\varphi \in C_0^\infty(\mathbb{R})$  is a bump function satisfying (2.6), (2.7),  $V \in L_\gamma^1(\mathbb{R})$ ,  $\gamma = 1 + s$ ,  $s \in (0, 1)$ , then for  $M \in (0, \infty)$  the filtered Fourier transform of  $a^\pm(x, \tau)b^\pm(y, \tau)$  satisfies the pointwise estimate:*

$$\mathbb{1}_{\pm x > 0} \mathbb{1}_{\pm y > 0} |\mathcal{F}_{\varphi, M}(a^\pm(x, \cdot)b^\pm(y, \cdot))(\xi)| \leq f_1(x)f_2^{(M)}(\xi)f_3(y), \quad (5.13)$$

where

$$f_1, f_3 \in L^{1/s, \infty}(\mathbb{R}) \cap L^\infty(\mathbb{R}), \quad f_2^{(M)}(\xi) \in L^1(\mathbb{R}), \quad \|f_2^{(M)}\|_{L^1(\mathbb{R})} \leq \frac{C}{\langle M \rangle}$$

with some constant  $C > 0$  independent of  $M$ .

Now we can proceed with the proof of Lemma 3.1.

*Proof of Lemma 3.1.* To fix the idea and to simplify the notation we consider the case involving  $b^+(y, \tau) = b(y, \tau)$ . We separate two cases:  $M \in (0, 1]$  and  $M \geq 1$ . For  $M \in (0, 1]$  our first step is to prove

$$\left\| \int_{\mathbb{R}} \mathbb{1}_{y > 0} \mathcal{F}_{\varphi, M}(b(y, \cdot))(x \pm y) f(y) dy \right\|_{L_x^p(\mathbb{R})} \leq C \|f\|_{L^q(\mathbb{R})}. \quad (5.14)$$

We use the pointwise estimate (5.12) so we can write

$$\mathbb{1}_{y > 0} |\mathcal{F}_{\varphi, M}(b(y, \cdot))(x \pm y)| \leq B_1^{(M)}(x \pm y) B_2(y),$$

where

$$B_1^{(M)} \in L^1(\mathbb{R}), \quad \|B_1^{(M)}\|_{L^1(\mathbb{R})} \leq C, \quad B_2 \in L^{1/s, \infty}(\mathbb{R})$$

and (5.14) follows from Young inequality

$$\left\| B_1^{(M)} * (B_2 f) \right\|_{L_x^p(\mathbb{R})} \leq C \|B_1^{(M)}\|_{L^1(\mathbb{R})} \|B_2 f\|_{L^p(\mathbb{R})}, \quad (5.15)$$

and the Hölder estimate

$$\|B_2 f\|_{L^p(\mathbb{R})} \leq C \|f\|_{L^q(\mathbb{R})}, \quad B_2 \in L^{1/s, \infty}(\mathbb{R}), \quad \frac{1}{q} = \frac{1}{p} - s. \quad (5.16)$$

Similarly, to prove

$$\left\| \int_{\mathbb{R}} \mathbb{1}_{x > 0} \mathcal{F}_{\varphi, M}(b(x, \cdot))(x \pm y) f(y) dy \right\|_{L_x^p(\mathbb{R})} \leq C \|f\|_{L^q(\mathbb{R})} \quad (5.17)$$

we use the pointwise estimate (5.12) again, so we can write

$$\mathbb{1}_{x > 0} |\mathcal{F}_{\varphi, M}(b(x, \cdot))(x \pm y)| \leq B_1^{(M)}(x \pm y) B_2(x),$$

where

$$B_1^{(M)} \in L^1(\mathbb{R}), \quad \|B_1^{(M)}\|_{L^1(\mathbb{R})} \leq C, \quad B_2 \in L^{1/s, \infty}(\mathbb{R}).$$

This time we have to estimate the term

$$\left\| B_2(B_1^{(M)} * f) \right\|_{L_x^p(\mathbb{R})}$$

so first we apply Hölder estimate (5.16) and then the Young convolution inequality.

Finally, the estimate (3.9) follows from (5.13) since we have

$$\mathbb{1}_{x > 0} \mathbb{1}_{y > 0} |\mathcal{F}_{\varphi, M}(b_1(x, \cdot)b_2(y, \cdot))(x \pm y)| \leq B_1^{(M)}(x \pm y) B_2(y) B_3(x),$$

where

$$B_1^{(M)} \in L^1(\mathbb{R}), \quad \|B_1^{(M)}\|_{L^1(\mathbb{R})} \leq C, \quad B_2(y), B_3(x) \in L^{1/s, \infty}(\mathbb{R}) \cap L^\infty(\mathbb{R}).$$

This completes the proof for the case  $M \in (0, 1]$ . For  $M \geq 1$  we simply use the fact that we have better estimate

$$\|B_1^{(M)}\|_{L^1(\mathbb{R})} \leq CM^{-1}$$

and we prove (3.8) and (3.9) assuming  $V \in L_1^1(\mathbb{R})$  only. This completes the proof.  $\square$

## 6 Equivalence of homogeneous Sobolev norms

In this section we are going to prove Lemma 1.

*Proof of the inequality (3.11).* The relation (3.6) guarantees that

$$\pi_k^{ac}(f)(x) - I_k(f)(x)$$

can be represented as a sum of remainder terms of the form

$$\begin{aligned} & \sum_{\epsilon_1, \dots, \epsilon_4 = \pm 1} \mathbf{1}_{\epsilon_1 x > 0} \int_{\mathbb{R}} \mathbf{1}_{\epsilon_2 y > 0} \mathcal{F}_{\varphi, M}(b_1(x, \cdot))(\epsilon_3 x + \epsilon_4 y) f(y) dy + \\ & + \sum_{\epsilon_1, \dots, \epsilon_4 = \pm 1} \mathbf{1}_{\epsilon_1 x > 0} \int_{\mathbb{R}} \mathbf{1}_{\epsilon_2 y > 0} \mathcal{F}_{\varphi, M}(b_2(y, \cdot))(\epsilon_3 x + \epsilon_4 y) f(y) dy + \\ & + \sum_{\epsilon_1, \dots, \epsilon_4 = \pm 1} \mathbf{1}_{\epsilon_1 x > 0} \int_{\mathbb{R}} \mathbf{1}_{\epsilon_2 y > 0} \mathcal{F}_{\varphi, M}(b_3(x, \cdot) b_4(y, \cdot))(\epsilon_3 x + \epsilon_4 y) f(y) dy, \end{aligned}$$

such that the estimates of Lemma 3.1 imply

$$\|(\pi_k - I_k) f\|_{L^p(\mathbb{R})} \leq \frac{C}{\langle 2^k \rangle} \|f\|_{L^q(\mathbb{R})},$$

with

$$\frac{1}{q} = \frac{1}{p} - s.$$

Using the inequalities

$$\begin{aligned} & \left\| \|2^{ks} (\pi_k - I_k) f\|_{\ell_k^2} \right\|_{L_x^p(\mathbb{R})} \leq \left\| \|2^{ks} (\pi_k - I_k) f\|_{\ell_k^1} \right\|_{L_x^p(\mathbb{R})} \leq \\ & \leq \left\| \|2^{ks} (\pi_k - I_k) f\|_{L_x^p(\mathbb{R})} \right\|_{\ell_k^1} \leq \left\| \frac{2^{ks}}{\langle 2^k \rangle} \right\|_{\ell_k^1} \|f\|_{L_x^q(\mathbb{R})}, \end{aligned}$$

and so we deduce (3.11).

This completes the proof.  $\square$

*Proof of Lemma 1.* Our main goal is to establish the following estimate

$$\left\| \|2^{ks} (\pi_k - \pi_k^0) f\|_{\ell_k^2} \right\|_{L^p(\mathbb{R})} \leq C \|f\|_{L^q(\mathbb{R})}, \quad (6.1)$$

with  $1/q = 1/p - s$ .

We start proving that

$$\left\| \|2^{ks} (\pi_k - \pi_k^0) f\|_{\ell_{k \leq 0}^2} \right\|_{L^p(\mathbb{R})} \leq C \|f\|_{L^q(\mathbb{R})}. \quad (6.2)$$

In particular, it will be enough to prove the inequality (3.12), i.e.

$$\left\| \|2^{ks} (I_k - \pi_k^0) f\|_{\ell_{k \leq 0}^2} \right\|_{L^p(\mathbb{R})} \leq C \|f\|_{L^q(\mathbb{R})},$$

since the estimate (3.11) has been just established above.

Using the decomposition

$$f = \sum_{j \in \mathbb{Z}} f_j^0,$$

we have that

$$(I_k - \pi_k^0) f = (I_k - \pi_k^0) f_{k-2, k+2}^0. \quad (6.3)$$

Indeed, it follows from

$$(I_k - \pi_k^0) f_{\leq k-2}^0(x) = \int \int e^{i(x+y)\tau} \varphi\left(\frac{\tau}{2^k}\right) f_{\leq k-2}^0(y) d\tau dy = 0$$

and

$$(I_k - \pi_k^0) f_{\geq k-2}^0(x) = \int \int e^{i(x+y)\tau} \varphi\left(\frac{\tau}{2^k}\right) f_{\geq k-2}^0(y) d\tau dy = 0.$$

Moreover, the expression of the leading term shows that the kernel  $(I_k - \pi_k^0)(x, y)$  can be represented as sum of the terms

$$\mathbb{1}_{\epsilon_1 x > 0} \mathbb{1}_{\epsilon_2 y > 0} \mathcal{F}_{\varphi, M}(a)(\epsilon_3 x + \epsilon_4 y),$$

with  $\epsilon_j = \pm 1, j = 1, \dots, 4$ ,  $\epsilon_1 \epsilon_2 \epsilon_3 \epsilon_4 = 1$  and  $a \in \mathcal{A}$ , defined in (3.2).

For simplicity we consider the case  $a = 1$ ,  $\epsilon_j = 1, \forall j = 1, \dots, 4$ , and we shall estimate the term

$$\int \mathbb{1}_{x > 0} \mathbb{1}_{y > 0} e^{i\tau(x+y)} \varphi\left(\frac{\tau}{M}\right) d\tau.$$

Then, we can proceed similarly for the other terms.

Integrating by parts and using Lemma 4.2, we get

$$\begin{aligned} \left\| 2^{ks} \int \int \mathbb{1}_{x > 0} \mathbb{1}_{y > 0} e^{i\tau(x+y)} \varphi\left(\frac{\tau}{2^k}\right) f_k^0(y) d\tau dy \right\|_{\ell_{k \leq 0}^2} &\leq \\ &\leq C \int \left\| \frac{2^{k(s+1)} \mathbb{1}_{x > 0} \mathbb{1}_{y > 0}}{\langle 2^k(x+y) \rangle^{1+s}} f_k^0(y) dy \right\|_{\ell_{k \leq 0}^2} dy \\ &\leq C \int \left\| \frac{2^{k(s+1)} \mathbb{1}_{x > 0} \mathbb{1}_{y > 0}}{\langle 2^k(x+y) \rangle^{1+s}} \right\|_{\ell_{k \leq 0}^\infty} \|f_k^0\|_{\ell_{k \leq 0}^2} dy. \end{aligned}$$

From the trivial inequality

$$\left\| \frac{2^{k(s+1)}}{\langle 2^k x \rangle^{1+s}} \right\|_{\ell_{k \leq 0}^\infty} \leq \frac{C}{|x|^{1+s}}$$

combined with the Young inequality in Lorentz spaces we have

$$\left\| \left\| 2^{ks} \int \int \mathbb{1}_{x > 0} \mathbb{1}_{y > 0} e^{i\tau(x+y)} \varphi\left(\frac{\tau}{2^k}\right) f_k^0(y) d\tau dy \right\|_{\ell_{k \leq 0}^2} \right\|_{L^p(\mathbb{R})} \leq C \left\| \|f_k^0(y)\|_{\ell_{k \leq 0}^2} \right\|_{L^q(\mathbb{R})},$$

with  $1/q = 1/p - s$  and  $0 < s < 1/p$ .

The case  $k \geq 0$  follows similarly using the estimate

$$|(\pi_k - \pi_k^0)f(x)| \leq C \int \frac{f(y)}{\langle 2^k(x \pm y) \rangle^s} \left( \frac{1}{\langle x \rangle} + \frac{1}{\langle y \rangle} \right) dy.$$

This complete the proof.  $\square$

## 7 Counterexample for equivalence of homogeneous Sobolev spaces

In this section we consider the case  $p \in [n/2, \infty) \cap (1, \infty)$  and we shall prove Theorem 1, therefore we shall show that the equivalence property

$$\|(\mathcal{H}_0 + V)^{n/(2p)} u\|_{L^p(\mathbb{R}^n)} \sim \|(\mathcal{H}_0)^{n/(2p)} u\|_{L^p(\mathbb{R}^n)} \quad (7.1)$$

is not true for  $n \in \mathbb{N}$ .

*Proof of Theorem 1.* Let us suppose that the relation (7.1) holds. Choosing positive potential

$$V(x) = \frac{1}{1 + |x|^3},$$

we can apply the heat kernel estimate obtained in [20], i.e.

$$\frac{C_1 e^{-c_1 |x-y|^2/4t}}{t^{n/2}} \leq e^{-t\mathcal{H}}(x, y) \leq \frac{C_2 e^{-c_2 |x-y|^2/4t}}{t^{n/2}}. \quad (7.2)$$

This estimate and the relation

$$\mathcal{H}^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} e^{-t\mathcal{H}} dt$$

imply

$$|(\mathcal{H}_0 + V)^{-1}u(x)| \leq C |(\mathcal{H}_0)^{-1}u(x)|$$

so taking the  $L^p$  norm and using a duality argument, we can write

$$\|V(\mathcal{H}_0 + V)^{-1}f\|_{L^p(\mathbb{R}^n)} \leq C \|f\|_{L^p(\mathbb{R}^n)}, \quad (7.3)$$

so we have

$$\|Vg\|_{L^p(\mathbb{R}^n)} \leq C \|(\mathcal{H}_0 + V)g\|_{L^p(\mathbb{R}^n)}. \quad (7.4)$$

Interpolation argument and the assumption  $p \geq n/2$  combined with the equivalence property (7.1) lead to

$$\int_{\mathbb{R}^n} (V(x))^{n/2} |u(x)|^p dx \leq C \|\mathcal{H}_0^{n/(2p)} u\|_{L^p(\mathbb{R}^n)}^p. \quad (7.5)$$

Taking  $u$  in the Schwartz class  $S(\mathbb{R}^n)$  of rapidly decreasing function, we can apply a rescaling argument. Indeed, considering the dilation

$$u_\lambda(x) = u(x\lambda),$$

we find

$$\|\mathcal{H}_0^{n/(2p)} u_\lambda\|_{L^p(\mathbb{R}^n)}^p = \underbrace{\|\mathcal{H}_0^{n/(2p)} u\|_{L^2(\mathbb{R}^n)}^p}_{\text{constant in } \lambda}$$

and

$$\lim_{\lambda \searrow 0} \int_{\mathbb{R}^n} V^{n/2}(x) |u_\lambda(x)|^p dx = \left( \int_{\mathbb{R}^n} V^{n/2}(x) dx \right) |u(0)|^p.$$

In this way we deduce

$$|u(0)|^p \left( \int_{\mathbb{R}^n} V^{n/2}(x) dx \right) \leq C \|\mathcal{H}_0^{n/(2p)} u\|_{L^p(\mathbb{R}^n)}^p. \quad (7.6)$$

The homogeneous norm

$$\|\mathcal{H}_0^{n/(2p)} u\|_{L^p(\mathbb{R}^n)}^p$$

is also invariant under translations, i.e. setting

$$u^{(\tau)}(x) = u(x + \tau),$$

we have

$$\widehat{u^{(\tau)}}(\xi) = e^{i\tau\xi} \widehat{u}(\xi)$$

and

$$\|\mathcal{H}_0^{n/(2p)} u^{(\tau)}\|_{L^p(\mathbb{R}^n)}^p = \|\mathcal{H}_0^{n/(2p)} u\|_{L^p(\mathbb{R}^n)}^p,$$

so applying (7.6) with  $u^{(\tau)}$  in the place of  $u$ , we find

$$|u(\tau)|^p \int_{\mathbb{R}^n} V^{n/2}(x) dx \leq C \|\mathcal{H}_0^{n/(2p)} u\|_{L^p(\mathbb{R}^n)}^p,$$



or equivalently

$$\|u\|_{L^\infty(\mathbb{R}^n)}^p \leq C_1 \|\mathcal{H}_0^{n/(2p)} u\|_{L^p(\mathbb{R}^n)}^p, \quad (7.7)$$

where

$$C_1 = \frac{C}{\|V^{n/2}\|_{L^1(\mathbb{R}^n)}}.$$

The substitution  $\phi = \mathcal{H}_0^{n/(2p)} u$  enables us to rewrite (7.7) as

$$\|I_{n/p}(\phi)\|_{L^\infty(\mathbb{R}^n)}^p \leq C_1 \|\phi\|_{L^p(\mathbb{R}^n)}^p, \quad (7.8)$$

where

$$I_\alpha(\phi)(x) = \mathcal{H}_0^{-\alpha/2}(\phi)(x) = c \int_{\mathbb{R}^n} |x-y|^{-n+\alpha} \phi(y) dy, \quad \alpha \in (0, n)$$

are the Riesz operators.

It is easy to show that (7.8) leads to a contradiction. Indeed, taking

$$\phi_N(x) = \sum_{j=0}^N \underbrace{|x|^{-n/p} \mathbb{1}_{2^j \leq |x| \leq 2^{j+1}}(x)}_{\chi_j(x)},$$

with  $N \geq 2$  sufficiently large and being  $\mathbb{1}_A(x)$  the characteristic function of the set  $A$ . Since the functions  $\chi_j$  have almost disjoint supports and they are non-negative, for almost every  $x \in \mathbb{R}$  we have

$$\sum_{j=1}^N \chi_j^p(x) = \left( \sum_{j=1}^N \chi_j(x) \right)^p.$$

so

$$\|\phi_N\|_{L^p(\mathbb{R}^n)}^p = \sum_{j=0}^N \int_{2^j}^{2^{j+1}} \frac{r^{n-1} dr}{r^n} \leq C' N.$$

Further, we can use the estimates

$$I_{n/p}(\phi_N)(0) \geq \left( \sum_{j=0}^N \int_{2^j}^{2^{j+1}} \frac{r^{n-1} dr}{r^n} \right) \geq CN.$$

Hence, from (7.8) we deduce

$$CN^p \leq \|I_{n/p}(\phi_N)\|_{L^\infty(\mathbb{R}^n)}^p \leq C_1 \|\phi_N\|_{L^p(\mathbb{R}^n)}^p \leq C_2 N,$$

for any  $N$  sufficiently big and this is impossible.

This completes the proof of the Theorem.  $\square$

## 8 Proof of Lemma 2

*Step I: Pseudo conformal two parameter group  $U(T, S)$ . Set*

$$\psi(t) = e^{-i(t-1)\mathcal{H}} f, t > 1, \quad (8.1)$$

where  $\mathcal{H}_0 = -\partial_x^2$ .

Making the transformation

$$(t, \psi) \implies (T, \Psi),$$

such that where

$$t = \frac{1}{T}, \quad \Psi(T, x) = \overline{\psi\left(\frac{1}{T}, x\right)}. \quad (8.2)$$

We can rewrite (8.1) as follows

$$\Psi(T) = e^{i(\mathcal{H}/T - \mathcal{H})} \bar{f}. \quad (8.3)$$

Now we can use the isometry

$$B(T) : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R}),$$

associated with the pseudo conformal transform for the free Schrödinger equation, i.e.

$$B(T) = M(T)\sigma_T, \quad (8.4)$$

with

$$M(T)g(x) = e^{ix^2/(4T)}g(x), \quad \sigma_T(g)(x) = T^{-1/2}g(T^{-1}x). \quad (8.5)$$

Making the substitution

$$\Phi(T) = B(T)\Psi(T),$$

we find the integral equation

$$\Phi(T) = U(T)U^*(1)\Phi_0, \quad \Phi_0 = B(1)(\bar{f}), \quad (8.6)$$

where

$$U(T) = B(T)e^{i\mathcal{H}/T} \quad (8.7)$$

We shall need the following properties of the two parameter group

$$U(T, S) = U(T)U^*(S) = B(T)e^{i(\mathcal{H}/T - \mathcal{H}/S)}B^*(S).$$

**Lemma 8.1.** *If*

$$-\Delta(T) = -\Delta + T^{-2}V\left(\frac{x}{T}\right), \quad (8.8)$$

*then for any  $T \in (0, 1]$  this operator is self - adjoint positive, we have the group property*

$$U(T_1, T_2)U(T_2, T_3) = U(T_1, T_3), \quad \forall T_1, T_2, T_3 \in (0, 1] \quad (8.9)$$

*and for any couple  $T, S \in (0, 1]$  we have*

$$U(T, S) : D((-\Delta(S))^{a/2}) \rightarrow D((-\Delta(T))^{a/2}), \quad \forall a \in [0, 2]. \quad (8.10)$$

Note that we have the relation

$$\|(t\partial_x + ix)\psi(t)\|_{L^2} \sim \|(-\Delta)^{1/2}\Phi(T)\|_{L^2}, \quad T = 1/t. \quad (8.11)$$

Hence the proof of Lemma 2 is reduced to the proof of the following estimate.

**Lemma 8.2.** *For any  $f \in S(\mathbb{R})$  with  $f(0) \neq 0$  we have*

$$\limsup_{T \searrow 0} \|\Phi(T)\|_{H^1(\mathbb{R})} = \infty.$$

□

*Step II: Proof of Lemma 8.2.* We shall argue by contradiction. If the assertion of the Theorem is not true then we can find  $C > 0$  so that

$$\|\Phi(T)\|_{H^1(\mathbb{R})} \leq C\|\Phi_0\|_{H^1(\mathbb{R})}, \quad \forall T \in (0, 1]. \quad (8.12)$$

The two parameter group  $U(T, S)$  has the property

$$U(T, S) : D((-\Delta(S))^{a/2}) \rightarrow D((-\Delta(T))^{a/2}), \quad \forall a \in [0, 2]. \quad (8.13)$$

and this means that we have in particular the inequality

$$\|(-\Delta(T))^{1/2} \underbrace{U(T, 1)\Phi_0}_{\Phi(T)}\|_{L^2(\mathbb{R})} \leq C\|(1 - \Delta(1))^{1/2}\Phi_0\|_{L^2(\mathbb{R})} \leq C\|\Phi_0\|_{H^1(\mathbb{R})}, \quad (8.14)$$

since we assume  $V \in L^\infty(\mathbb{R})$ . The property (8.12) implies now

$$\|(-\Delta)^{1/2} \underbrace{U(T, 1)\Phi_0}_{\Phi(T)}\|_{L^2(\mathbb{R})} \leq C\|\Phi_0\|_{H^1(\mathbb{R})}, \quad (8.15)$$

so using the relation

$$\|(-\Delta(T))^{1/2}\Phi(T)\|_{L^2(\mathbb{R})}^2 = \|(-\Delta)^{1/2}\Phi(T)\|_{L^2(\mathbb{R})}^2 + T^{-2} \int V(x/T)|\Phi(T, x)|^2 dx, \quad (8.16)$$

we get

$$T^{-2} \int V(x/T)|\Phi(T, x)|^2 dx \leq C\|\Phi_0\|_{H^1(\mathbb{R})}^2, \quad (8.17)$$

This is equivalent to the relation

$$\int V(y)|\phi(T, yT)|^2 dy \leq CT\|\Phi_0\|_{H^1(\mathbb{R})}^2, \quad (8.18)$$

so using the assumption  $\int V(y)dy \neq 0$  and taking the limit  $T \rightarrow 0$ , we get

$$\Phi_0(0) = f(0) = 0.$$

This is a contradiction and the proof of the Lemma is complete.  $\square$

## 9 Modified Lax pairs relations

Given any two different perturbed groups  $U(T, S)$ ,  $\tilde{U}(T, S)$  connected via the splitting relation

$$U(T, S) = B(T)\tilde{U}(T, S)B^*(S), \quad 0 < T, S \leq 1 \quad (9.1)$$

the corresponding time dependent generators  $-iH(T)$  and  $-i\tilde{H}(T)$  are determined by the Cauchy problems

$$\frac{d}{dT}U(T, S) = -iH(T)U(T, S), \quad U(S, S) = I. \quad (9.2)$$

$$\frac{d}{dT}\tilde{U}(T, S) = -i\tilde{H}(T)\tilde{U}(T, S), \quad \tilde{U}(S, S) = I. \quad (9.3)$$

Now (9.1) can be associated with the following Lax pairs relation

$$B'(T) = i \left[ B(T)\tilde{H}(T) - H(T)B(T) \right] \quad (9.4)$$

and we can easily see that (9.4) implies that  $-iH(T)$  is the generator of the perturbed group  $U(T, S)$ .

Now we apply this argument for

$$U_0(T, S) = B(T)\tilde{U}_0(T, S)B^*(S), \quad 0 < T, S \leq 1 \quad (9.5)$$

with

$$U_0(T, S) = e^{-i\mathcal{H}_0(T-S)} = U_0(T)U_0^*(S), \quad \tilde{U}_0(T, S) = e^{i\mathcal{H}_0/T}e^{-i\mathcal{H}_0/S}.$$

Obviously, the generator of  $U_0(T, S)$  is  $-i\mathcal{H}_0 = i\partial_x^2$  and the Lax pairs relation becomes now

$$B'(T) = i \left[ B(T)\frac{\mathcal{H}_0}{T^2} - \mathcal{H}_0 B(T) \right] \quad (9.6)$$

The check of this relation is straightforward and we omit it.

Now we can define the family of operators

$$U_0(T) = B(T)e^{i\mathcal{H}_0/T} \quad (9.7)$$

This relation and the definition of  $B(T)$  imply

$$U_0(T) = e^{-i\mathcal{H}_0 T}. \quad (9.8)$$

Further the perturbed group  $U(T, S)$  defined by (8.7) is of the form introduced in (9.1) with

$$\tilde{U}(T, S) = e^{i\mathcal{H}/T} e^{-i\mathcal{H}/S}$$

and obviously the generator of  $\tilde{U}(T, S)$  is  $-i\tilde{H}(T) = -i\mathcal{H}/T^2$ . The modified Lax pairs relation has the form

$$B'(T) = i \left[ B(T) \frac{\mathcal{H}}{T^2} - H(T) B(T) \right] \quad (9.9)$$

and this relation is true with

$$H(T) = -\Delta(T),$$

where

$$-\Delta(T) = T^{-2} \sigma_T \mathcal{H} \sigma_T^* = -\Delta + T^{-2} V \left( \frac{x}{T} \right). \quad (9.10)$$

Again the check of the relation is trivial consequence of (9.6) and we omit the details.

## Acknowledgements

The authors are grateful to Atanas Stefanov for the critical remarks and discussions during the preparation of the work.

Funding: This work was supported by by University of Pisa, project no. PRA-2016-41 Fenomeni singolari in problemi deterministici e stocastici ed applicazioni; by INDAM, GNAMPA - Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni and by Institute of Mathematics and Informatics, Bulgarian Academy of Sciences.

## References

- [1] Artbazar G., K. Yajima K.: The  $L^p$  - continuity of wave operators for one dimensional Schrödinger operators. J. Math. Sci. Univ. Tokyo **7**(2), 221 – 240 (2000)
- [2] Cuccagna S., Georgiev V., Visciglia N.: Decay and scattering of small solutions of pure power NLS in  $\mathbb{R}$  with  $p > 3$  and with a potential. Comm. Pure Appl. Math. **67**(6), 957 – 981 (2014)
- [3] D'Ancona P., Fanelli L.:  $L^p$ -Boundedness of the Wave Operator for the One Dimensional Schrödinger Operator, Commun. Math. Phys. **268**, 415 - 438 (2006)
- [4] Deift P., Trubowitz E.: Inverse scattering on the line. Comm. Pure Appl. Math. **32**, 121 - 251 (1979).
- [5] Georgiev V., Giammetta A. R.: Sectorial Hamiltonians without zero resonance in one dimension. In: Recent Advances in Partial Differential Equations and applications, vol. 666 of Contemporary Mathematics , 225-238 AMS (2016)
- [6] Georgiev V., Giammetta A. R.: On homogeneous Besov spaces for 1D Hamiltonians without zero resonance, (2016) arXiv:1605.02581
- [7] Georgiev V., Visciglia N.: Decay estimates for the wave equation with potential. Comm. Part. Diff. Eq. **28** (7,8), 1325 - 1369 (2003).
- [8] Grafakos L., Maldonado D., Naibo V.: A remark on an endpoint Kato-Ponce inequality. Differential Integral Equations. **27**, 415 - 424 (2014)

- [9] Grafakos L., Si Z.: The Hörmander multiplier theorem for multilinear operators. J. Reine Angew. Math. **668**, 133 - 147 (2012)
- [10] Gröchenig K., Klotz A.: Norm-controlled inversion in smooth Banach algebras. J. Lond. Math. Soc. **88** (2) 49 - 64 (2013)
- [11] Henry D.: *Geometric Theory of Semilinear Parabolic Equations. Series: Lecture Notes in Mathematics*, **840**, Springer (2009)
- [12] Hörmander L. *The Analysis of Linear Partial Differential Operators, vol. II, Differential Operators with Constant Coefficients*, Springer, Berlin, (2005)
- [13] Kato T., Ponce G.: Commutator estimates and the Euler and Navier-Stokes equations. Comm. Pure Appl. Math. **41**, 891 - 907 (1988)
- [14] Kenig C.E., Ponce G., Vega L.: Well-posedness and scattering results for the generalized Korteweg-de Vries equation via the contraction principle. Comm. Pure Appl. Math. **46**, 527 - 620 (1993)
- [15] Lax P., Phillips R.: Scattering theory. Bull. Amer. Math. Soc. **70** (1), 130 - 142 (1964)
- [16] Reed M., Simon B.: *Methods of Modern Mathematical Physics, Vol. III: Scattering Theory* Academic Press, (1978)
- [17] Weder R.: The  $W_{k;p}$  - Continuity of the Schrödinger Wave Operators on the Line. Commun. Math. Phys. **208** 507 - 520 (1999)
- [18] Weder R.:  $L^p - L^{p'}$  Estimates for the Schrödinger Equation on the Line and Inverse Scattering for the Nonlinear Schrödinger Equation with a Potential. Journal of Functional Analysis. **170** 37 - 68 (2000)
- [19] Yajima K.: The  $W^{k;p}$ -continuity of wave operators for Schrodinger operators. J. Math. Soc. Japan. **47** 551 - 581 (1995)
- [20] Zhang Q. S.: Large Time Behavior of Schrödinger Heat Kernels and Applications. Commun. Math. Phys. **210**, 371 - 398 (2000)
- [21] Zheng A.: Spectral multipliers for Schrödinger operators. Illinois Journal of Mathematics. **54** (2), 621 - 647 (2010)